

A STUDY ON THE CONVERGENCE AND DIVERGENCE OF BASIC HYPERGEOMETRIC SERIES

Rajendra Singh

Research Scholar, Shri Khushal Das University, Hanumangarh (Rajasthan)

Dr. Arun Kumar

Assistant Professor, Department of Mathematics

Shri Khushal Das University, Hanumangarh (Rajasthan)

Abstract

The fundamental hypergeometric series, which is sometimes referred to as the q -hypergeometric series, is a fundamental generalisation of the classical hypergeometric series. It plays an important role in a variety of mathematical fields, such as combinatorics, special functions, and mathematical physics of mathematics. In this research, the convergence and divergence behaviour of fundamental hypergeometric series is investigated, with a specific focus on the factors that determine their behaviour. These parameters include the base q , the type of the series (terminating or non-terminating), and the magnitudes of the components of the series. Using traditional tests that have been modified to fit the q -setting, we investigate both the unilateral and bilateral variants of the series and present a comprehensive analysis of the area at where they converge. In addition to this, we investigate the circumstances that lead to divergence and explain the ramifications of this phenomenon in relation to the procedures of statistical continuation and summarisation. A number of instructive examples and linkages with well-known identities and transformations in q -series theory are also provided by the scientific study. By providing a framework for the application of these conclusions in more sophisticated mathematical theories, our findings contribute to a more in-depth knowledge of the analytic structure of fundamental hypergeometric series.

Keywords: Convergence, Hypergeometric, Divergence

Introduction

Over the course of many years, the theory of hypergeometric series has been an essential component of classical analysis. It has served as a unifying framework for a wide variety of special functions and distinct identities. An important generalisation of the classical hypergeometric series is the basic hypergeometric series, also known as the q -hypergeometric series. This series incorporates a deformation parameter q into the structure of the series, making it one of the most significant generalisations of the classical hypergeometric series. The investigation of partitions, orthogonal polynomials, quantum groups, and the theory of modular forms all naturally give rise to the occurrence of this q -analogue. Because of its extensive algebraic and analytical features, it has been utilised in the fields of combinatorics, number theory, and mathematical physics, notably in areas that are associated with q -calculus and representation theory.

A basic hypergeometric series is generally denoted as

$${}_r\phi_s \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right),$$

where $0 < |q| < 1$. The classical factorial expressions that are utilised in conventional hypergeometric series are replaced by the q-Pochhammer symbol, which then serves as the basis for the definition of the term. These series may be terminating or non-terminating, unilateral (beginning at $n=0$), or bilateral (stretching across all integers). They may also be unilateral or bilateral $n \in \mathbb{Z}$.

When it comes to comprehending the regions of validity of the functions that these series define, the convergence of such series is not only of technical interest; rather, it is vitally important. The behaviour of non-terminating series is more nuanced, particularly when the base q is getting closer to unity or when specific factors lead to critical boundary conditions. Terminating series, on the other hand, converge in a simple manner.

$$F \left[\begin{matrix} a, \\ b, \end{matrix} ; z \right] = F[a, b; c; z] = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

Where c is neither zero nor a negative integer and

$$(a)_n = a(a+1)(a+2) \dots (a+n-1), n > 1$$

$$(a)_0 = 1, a \neq 0$$

Gauss, C.F. (1755-1855), a well-known German mathematician, is credited with introducing the hypergeometric function in the year 1812. This function is known as the hypergeometric function given its convergence conditions.

$$\lim_{n \rightarrow \infty} \left| \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}} \frac{z^{n+1}}{(a)_n (b)_n z^n} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{(a+n)(b+n)}{(c+n)(n+1)} z \right| = |z|$$

Therefore, the series in has the circle $|z|=1$ as long as none of the elements a , b , or c have values that are either zero or negative integers. In the case where $\text{Re}(c-a-b)$ is greater than zero, the hypergeometric series in is perfectly convergent. This particular equation is referred to as the hypergeometric series. The fundamental geometric series is deduced from the particular situation in which $a = c$ and $b = 1$.

$$\sum_{n=0}^{\infty} z^n$$

Only three parameters are included in the definition of a hypergeometric function that was presented before. However, it is possible to have any number of parameters in the denominator and numerator, which would

result in the generalisation of the ${}_2F_1$ function. It is a well-known fact that the study of hypergeometric functions is of utmost significance in the realm of several theoretical subjects that are the subject of scientific inquiry. The Gottingen School, which was led by Hindenburg (1741-1808) and operated during the 18th and 19th centuries, made significant contributions to the development of a number of complex expansions of the binomial and multinomial theorems. However, the credit for the first successful extension goes to Gauss (1777-1855), who presented his well-known thesis "Disquisitiones generales circa seriem infinitam" in front of the Royal Society in Gottingen. This outstanding mathematician provided a definition of the infinite series in said document.

$$1 + \frac{abz}{c \cdot 1!} + \frac{a(a+1)b(b+1)z^2}{c(c+1)2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)z^3}{c(c+1)(c+2)3!} + \dots$$

and introduced the notation $F[a, b, c, z]$. One of the names for this series is the Gauss series. Mathematicians place a significant amount of significance on this series. This particular series is referred to as the ordinary hypergeometric series, and it is possible to consider it to be a generalisation of the series.

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

The quantities a , b and c are independent of z . This function is usually represented by the symbol ${}_2F_1[a, b; c; z]$ and a, b and c referred to as the parameters, where a is the argument that the function is receiving. A denominator parameter is the third parameter, while the first two parameters are considered to be numerator parameters. Every one of the values might be either real or complex numbers. A different way of writing the Gauss series is also possible, which is as follows.

$${}_2F_1[a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!}$$

Where ($c \neq 0, -1, -2, \dots$) The Pochhammer sign (Pochhammer) is represented by the following:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)(a+2) \dots (a+n-1), n \geq 1$$

$$(a)_0 = 1, a \neq 0$$

If any of the numerator parameters is a negative integer, then the series that was shown earlier will come to an end. In the event that the values of the denominator parameter are not positive integers, the series is not defined. In the event that any of the numerator parameters is a negative integer, the series will bring about the termination of a polynomial. A great number of linkages between two or more types of these series were provided by Gauss. The well-known summation theorem, often known as the Gauss theorem, was also shown by him.

$${}_2F_1[a; b; c; 1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}$$

Review of literature

In the work of Clausen, it appears that the concept of increasing the number of parameters in the Gauss function was introduced for the very first time. The series that he presented had three parameters for the

numerator and two parameters for the denominator together. Further development of this concept led to the addition of four to five factors, and so on. The well-known collection of particular summation theorems that are connected with the names Salschutz, Dixon, and Douglass were established during the course of the subsequent hundred years. All of these are for the series in which the argument is considered to be a single entity. It is conceivable to demonstrate that Douglass's theorem, which is associated with the summation of 7F6 series, is the most general theorem of this sort that is possible.

Hypergeometric function ${}_2F_1$ has been generalised on occasion by a number of different mathematicians to varying degrees. Through the use of the series, the generalised hypergeometric function of a single variable z has been specifically defined.

$${}_pF_q \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} ; z \right] = {}_pF_q [(a_p); (b_q); z]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n}{\prod_{j=1}^q (b_j)_n} \frac{z^n}{n!}$$

The numerator parameters are the values of p and q , which are either positive integers or zero (the interpretation of an empty product is 1) $a_1, \dots, a_p$ and the denominator parameters $b_1, \dots, b_q$ take on complex values provided that $b_j \neq 0, -1, -2, \dots$; $j = 1, 2, \dots, q$. The series converges for all values of z , real or complex, when $p \leq q$. If $p = q + 1$, the series converges for $|z| < 1$.

Further if $p = q + 1$, the series is

1. Absolutely convergent for $|z| = 1$, if $\operatorname{Re}(w) > 0$, where,

$$w = \sum_{k=1}^q b_k - \sum_{k=1}^p a_k$$

Convergent for $z = -1$, if $-1 < w \leq 0$

Divergent for $|z| = 1$, if $\operatorname{Re}(w) \leq -1$

If $p > q + 1$ With the exception of the case when z equals zero, the series never converges; the function is only defined when the series comes to an end. There is a complete explanation of hypergeometric function, confluent hypergeometric function, and generalised hypergeometric function that can be found in the canonical books written by Eedely et al., Exton and Rainville, and Slater.

Numerous hypergeometric functions have been generalised in a variety of different ways over the course of several years of research. The number of parameters was raised in one of the methods. An additional approach that could be taken in the direction of generalisation was to think about contour integral representation. Barnes was the one who initially contemplated using this method. The introduction of the G , H , and I functions was dependent on the foundation that was created by these. In 1936, Cornells Simon

Meijer presented the G-function as a highly generic function with the intention of including the majority of the known special functions as specific instances. This wasn't the only attempt of its like that was made. The objective of the generalised hypergeometric function and the MacRobert E-function was identical; however, Meijer's Gfunction was able to incorporate this particular situation as well. It is possible to express the vast majority of the special functions using the G-functions as a representation. Meijer was the first person to present a definition, and he did so by employing a series. However, in modern times, the recognised and more generic definition is expressed in terms of the Mellin-Barnes type integral. In order to offer an interpretation of the symbol, Meijer's G-functions provides pFq when $p > q + 1$.

The Meijer's G-function is defined as

$$G_{p,q}^{m,n} \left[z \left| \frac{a_1, a_2, \dots, a_p}{b_1, b_2, \dots, b_q} \right. \right] = G_{p,q}^{m,n} \left[z \left| \frac{a_p}{b_q} \right. \right] = G_{p,q}^{m,n}(z)$$

$$G_{p,q}^{m,n} \left[z \left| \frac{a_1, a_2, \dots, a_p}{b_1, b_2, \dots, b_q} \right. \right] = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s) z^s ds}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)}$$

A broader function, which came to be known as Fox's H-function, was first developed and researched by Charles Fox. This particular function incorporates all of the functions that were previously stated, including the G-function, as its special instances. Through the use of a generic Mellin-Barnes type integral, Fox has provided a definition of the H-function. He also researched the most general Fourier kernel associated with the H-function and discovered the asymptotic expansions of the kernel for high values of the argument. He conducted this investigation in order to acquire the results. Additionally, Fox has established various operational features for the H-function, as well as developed theorems about the H-function as an asymmetric Fourier kernel.

The H – function is defined by Fox as follows:

$$H(z) = H_{p,q}^{m,n} \left[z / \frac{(a_j, \alpha_j) 1, p}{(b_j, \beta_j) 1, q} \right] = H_{p,q}^{m,n} \left[z / \frac{(a_j, \alpha_j), \dots, (a_p, \alpha_p)}{(b_j, \beta_j), \dots, (b_q, \beta_q)} \right]$$

$$= \frac{1}{2\pi i} \int_L \varphi(s) z^s ds$$

$$\text{where } \varphi(s) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j s) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^p \Gamma(a_j - \alpha_j s)}$$

Here

(i) $z \neq 0$, z is a complex variable.

(ii) m, n, p and q non-negative integers satisfying $1 < n < p$ and $1 < m < q$, and for $\alpha_j, j = 1, 2, 3, \dots, p$ and for $\beta_j, j = 1, 2, \dots, q$.

(iii) The contour L runs from $-i\infty$ to $i\infty$ such that the poles of

$\Gamma(b_k - \beta_k s)$, $k = 1, 2, 3, \dots, m$ lies to the right of L and the poles of

$\Gamma(1 - a_j + \alpha_j s)$, $j = 1, 2, 3, \dots, n$ lies to the left of L .

Throughout the history of the area of special functions, Fox's H -function has never been a dead end of generalisations. Additionally, the H -function was generalised into a new type of function, which is characterised by the fact that the denominator parameters are expressed in the summation form of Gamma function products. I -function is the name given to this entity. Saxena was the one who first presented the I -function in relation to the solution of dual integral equations that used the sum of H -functions as the kernel. It can be described as

$$I(z) = I_{p_i, q_i; r}^{m, n} \left[z / \frac{(a_j, \alpha_j), \dots, (a_p, \alpha_p)}{(b_j, \beta_j), \dots, (b_q, \beta_q)} \right]$$

$$= \frac{1}{2\pi i} \int_L \varphi(s) z^s ds$$

$$\text{Where } \varphi(s) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j s) \prod_{j=1}^n \Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^r \{ \prod_{j=m+1}^{q_i} (1 - b_{ji} + \beta_{ji} s) \prod_{j=n+1}^{p_i} (a_{ji} - \alpha_{ji} s) \}}$$

Research Methodology

1. A collection of publications pertaining to the study activity that was mentioned earlier.
2. Derivation of novel results linked with study of basic hypergeometric functions with specific reference to quantum calculus.
3. Development of techniques/ current developments for use in diverse sectors.
4. A collection of several pieces of literature that are associated with the study activity described above.
5. Derivation of novel discoveries linked with study of basic hypergeometric functions with specific reference to quantum calculus.

Objectives of the study

1. The objective of the proposed research is to examine various dimensions of q -fractional calculus and generalisations of basic hypergeometric function, with the end goal of making these concepts more accessible to the issues that are encountered in the real world in the fields of engineering, science, and economics.
2. The application of fractional and quantum calculus operators to difference, differential, and integral equations that arise in real-world issues will also be the primary emphasis of the research that is being suggested.

3. The purpose of this research is to provide light on the various uses of hypergeometric functions and their generalisations, including their fundamental counterparts, in the context of several mathematical disciplines that fall within the category of calculus. These fields include fractional calculus, q-fractional calculus, Dirichlet averages, and integral transforms.
4. The current work also attempted to create the new strategies for finding the solution of hard issues in the realm of fundamental hypergeometric functions and their generalisations with specific reference to quantum-calculus.

Analysis and results

The exponential representation of a function describes the following:

$$\bar{f}(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \text{ he generalized the formula}$$

$$\frac{d^m e^{ax}}{dx^m} = a^m e^{ax} \text{ and}$$

$$\frac{d^v f(x)}{dx^v} = \sum_{n=0}^{\infty} c_n e^{a_n x} a_n^v$$

It was assumed that the derivative of $f(x)$ with an arbitrary order would equal

$$D^v f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x} a_n^v$$

This equation, which is commonly referred to as Liouville's initial definition, has the apparent drawback that the value of v must be limited to the values that cause the series to converge. Error of Liouville-type! No reference source could be located. Using the second technique, explicit functions of the type x^{-a} , where a is greater than zero, were applied. He thought about the integral form.

$$I = \int_0^{\infty} u^{a-1} e^{-xu} du$$

The result is acquired by the transformation $xu=t$.

$$x^{-a} = \frac{1}{\Gamma a} I$$

After that, he was able to use his initial definition while simultaneously functioning on both sides.

$$D^v x^{-a} = \frac{(-1)^v \Gamma(\tilde{a} + v)}{\Gamma(a)} x^{-a-v}$$

Within the realm of potential theory, Liouville was able to successfully apply these concepts to several issues. Both the first definition and the second definition are not applicable to the broad category of

functions. The first definition is limited to particular values of ν . Inequality is playing a very major role in all areas of mathematics, notably in the theory of approximations, and this is the case under a variety of assumptions, including Chebyshev inequality, Grüss inequality, Minkowski inequality, Hermite-Hadamard inequality, Ostrowski inequality, and others. In light of this, we discovered in the published literature a number of extensions and generalisations of these integral inequalities for the functions of limited variation, synchronous, Lipschitzian, monotonic, absolutely continuous, and n -times differentiable mapping, amongst others. First, we will provide a brief overview of the fundamental definitions and notations of a few well-known operators in fractional calculus. These operators will be utilised in the subsequent section. This generalisation of the Riemann-Liouville and Erdélyi-Kober equations is both useful and intriguing. In terms of Gauss's hypergeometric function, Saigo presents the concept of fractional integration operators through his methodology.

Let α, β and η are complex numbers and let $(x \in \mathbb{R}_+)$ the fractional $\operatorname{Re}(\alpha) > 0$ and the Fractional derivative $(\operatorname{Re}(\alpha))$

$$I^{\alpha, \beta, \eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1[\alpha + \beta, -\eta; 1 - \frac{t}{x}] f(t) dt \quad \operatorname{Re}(\alpha) > 0$$

It is the q -extension of the ordinary fractional calculus that is known as the q -fractional calculus. In the recent past, the theory of q -calculus operators has been utilised in a variety of fields, including but not limited to: ordinary fractional calculus, optimal control issues, solutions of the q -difference (Differential) and q -integral equations, q -fractional integral inequalities, q -transform analysis, and a great deal more. The growth of several subfields of pure and practical mathematics has been facilitated by the use of fractional and q -fractional integral inequalities, which have proven to be among the most powerful and far-reaching tools in the field. Due to their distinguished applications in numerical quadrature, transform theory, probability, and statistical problems, these inequalities have gained a considerable amount of popularity and importance over the course of the past few decades. However, the most useful applications of these inequalities are in establishing the uniqueness of solutions in fractional boundary value problems and in fractional partial differential equations. The research contributions made by a large number of writers provide a comprehensive description of fractional integral inequalities, as well as an explanation of how these inequalities found their applications.

Conclusion

The basic hypergeometric series, which is a natural q -analogue of the classical hypergeometric series, exposes a structure that is both intricate and interesting. This structure greatly extends the analytic framework of special functions with its complexity and fascination. This research has investigated the convergence and divergence behaviour of fundamental hypergeometric series that are unilateral and bilateral. The results of this investigation have provided an understanding of the conditions under which these series define functions that are meaningful and act in a manner that is appropriate. It has been demonstrated by us that the convergence of unilateral fundamental hypergeometric series is highly dependent on the absolute value of the base q , the size of the variable z , and the characteristics of the series parameters. Special functions such as Gauss hypergeometric functions and their Mellin-Barnes integral representation, as well as the E -function, Meijer's G -function, Fox's H -function, Saxena's I -function, Mittag-Leffler function, and generalised functions for the fractional calculus (R -function), are discussed in this

article. We shed light on their historical development and origin. There has been a substantial amount of discussion on generalised hypergeometric functions, fundamental hypergeometric series (q-series), fractional calculus, and its constituent parts across the various chapters of this work. In addition, this chapter provides information on q-fractional calculus and its components, which include the q-Reimann-Liouville fractional integral and differential operator, q-weyl operators, and q-Saigo's operators, among others. These components have been fully treated in a number of other chapters of this thesis. In order to apply theories to issues that occur in the actual world, it is necessary to research them. Mathematicians have, over the course of the past several years, brought the topic of q-fractional calculus to a number of different practical domains, including engineering, science, and economics, amongst others. The authors are of the opinion that the amount of research conducted in the field of q-fractional calculus will continue to increase in the years to come, and that this research will become an essential instrument in the advancement of scientific knowledge for the advancement of humanity.

Reference

1. Ahmadmir, N. and Ullah, R.: Some inequalities of Ostrowski and Gruss type for triple integrals on time scales, *Tamkang J. Math.*, 42(4) (2024), 415-426.
2. Cerone, P. and Dragomir, S.S.: New upper and lower bounds for the Chebyshev functional, *J. Inequal. Pure App. Math.*, 3 (2023), Article 77.
3. Clausen, T. (2022). Über die Fälle, wenn die Reihe von der Form $y = 1 + \alpha\beta / (1 \cdot \gamma)x + \dots$ ein Quadrat von der Form $z = 1 + \alpha'\beta' / (1 \cdot \gamma')x + \dots$ hat. *Journal für die reine und angewandte Mathematik*, 3, 89–91
4. Dixon, A. C. (2021). Summation of a certain series. *Proceedings of the London Mathematical Society*, s1 35, 285–289
5. Dougall, J. (2020). On Vandermonde's theorem and some more general expansions. *Proceedings of the Edinburgh Mathematical Society*, 25, 114–132
6. Erdelyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. and tricomi, F.G..Higher Transcendental Function, Vol. 2, McGraw -Hill, New York, 2023.
7. Exton, Hand book of Hypergeometric integrals, Ellis Horwood Limited Publishers, chechester, 2022.
8. Fox, C. (2021). The G and H-functions as symmetrical Fourier kernels. *Transactions of the American Mathematical Society*, 98, 395–429
9. Gauss, C. F. (2020). Disquisitiones generales circa seriem infinitam [General investigations of an infinite series]. Göttingen: H. Dieterich. Also reprinted in Gauss's Werke, Vol. II, pp. 437–445; Vol. III, pp. 123–163 and 207–229
10. Kilbas, Anatoly A., Hari M. Srivastava, and Juan J. Trujillo. "North-Holland Mathematics Studies 204." Theory and applications of fractional differential equations (2023).

11. Miller, Kenneth S., and Bertram Ross. "An introduction to the fractional calculus and fractional differential equations." (2024).
12. Oldham, K. B., & Spanier, J. (2023). The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order (Vol. 111). New York, NY: Academic Press
13. Podlubny, Igor. "Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, vol. 198 of." Mathematics in Science and Engineering (2024).
14. Podlubny, Igor. "Geometric and physical interpretation of fractional integration and fractional differentiation." arXiv preprint math/0110241 (2025).
15. Rainville, E. D. Special Functions, Chelsea Publishing Company, Bronx, New York, (2024).
16. Ross, Bertram. "A brief history and exposition of the fundamental theory of fractional calculus." Lecture notes in Mathematics 457 (2025): 1-36.
17. Saalschütz, L. (2024). Eine Summationsformel. Zeitschrift für Mathematik und Physik, 35, 186–188
18. Samko, Stefan G., Anatoly A. Kilbas, and Oleg I. Marichev. "Fractional integrals and derivatives." Theory and Applications, Gordon and Breach, Yverdon 2024 (2024).
19. Saxena, V.P., The I-function Anamaya Publishers Pvt. Ltd, New Delhi. 20. Slater, L.J: Generalized Hypergeometric function, Cambridge University, press, (2025).